

**What Is Claimed Is:**

1           1.       A method for using a computer system to solve an unconstrained  
2 interval global optimization problem specified by a function  $f$ , wherein  $f$  is a scalar  
3 function of a vector  $\mathbf{x} = (x_1, x_2, x_3, \dots x_n)$ , the method comprising:  
4           receiving a representation of the function  $f$  at the computer system;  
5           storing the representation in a memory within the computer system; and  
6           performing an interval global optimization process to compute guaranteed  
7 bounds on a globally minimum value of the function  $f(\mathbf{x})$  over a subbox  $\mathbf{X}$ ;  
8           wherein performing the interval global optimization process involves,  
9                   applying term consistency to a set of relations associated  
10                  with the function  $f$  over the subbox  $\mathbf{X}$ , and excluding any portion of  
11                  the subbox  $\mathbf{X}$  that violates any of these relations,  
12                  applying box consistency to the set of relations associated  
13                  with the function  $f$  over the subbox  $\mathbf{X}$ , and excluding any portion of  
14                  the subbox  $\mathbf{X}$  that violates any of these relations, and  
15                  performing an interval Newton step on the subbox  $\mathbf{X}$  to  
16                  produce a resulting subbox  $\mathbf{Y}$ , wherein the point of expansion of  
17                  the interval Newton step is a point  $\mathbf{x}$  within the subbox  $\mathbf{X}$ , and  
18                  wherein performing the interval Newton step involves evaluating  
19                  the gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$  using interval arithmetic.

1           2.       The method of claim 1, wherein applying term consistency  
2 involves:  
3           symbolically manipulating an equation within the computer system to  
4 solve for a term  $g(x_j)$ , thereby producing a modified equation  $g(x_j) = h(\mathbf{x})$ , wherein  
5 the term  $g(x_j)$  can be analytically inverted to produce an inverse function  $g^{-1}(y)$ ;

6 substituting the subbox  $\mathbf{X}$  into the modified equation to produce the  
7 equation  $g(X'_j) = h(\mathbf{X})$ ;  
8 solving for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and  
9 intersecting  $X'_j$  with the interval  $X_j$  to produce a new subbox  $\mathbf{X}^+$ ;  
10 wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the equation within  
11 the subbox  $\mathbf{X}$ , and wherein the size of the new subbox  $\mathbf{X}^+$  is less than or equal to  
12 the size of the subbox  $\mathbf{X}$ .

1 3. The method of claim 1, wherein performing the interval global  
2 optimization process involves:  
3 keeping track of a smallest upper bound  $f\_bar$  of the function  $f(\mathbf{x})$ ;  
4 removing from consideration any subbox  $\mathbf{X}$  for which  $f(\mathbf{X}) > f\_bar$ ; and  
5 wherein applying term consistency to the  $f\_bar$  relation involves applying  
6 term consistency to the inequality  $f(\mathbf{x}) \leq f\_bar$  over the subbox  $\mathbf{X}$ .

1 4. The method of claim 3, wherein applying box consistency to the  
2 set of relations involves applying box consistency to the inequality  $f(\mathbf{x}) \leq f\_bar$   
3 over the subbox  $\mathbf{X}$ .

1 5. The method of claim 1, wherein performing the interval global  
2 optimization process involves:  
3 determining the gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  includes  
4 components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ );  
5 removing from consideration any subbox for which any element of  $\mathbf{g}(\mathbf{x})$  is  
6 bounded away from zero, thereby indicating that the subbox does not include a  
7 stationary point of  $f(\mathbf{x})$ ; and

8            wherein applying term consistency to the set of relations involves applying  
9   term consistency to each component  $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$  over the subbox  
10  $\mathbf{X}$ .

1            6.        The method of claim 5, wherein applying box consistency to the  
2   set of relations involves applying box consistency to each component  
3    $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$  over the subbox  $\mathbf{X}$ .

1            7.        The method of claim 1, wherein performing the interval global  
2   optimization process involves:  
3              determining diagonal elements  $H_{ii}(\mathbf{x})$  ( $i=1, \dots, n$ ) of the Hessian of the  
4   function  $f(\mathbf{x})$ ;  
5              removing from consideration any subbox for which a diagonal element of  
6   the Hessian is always negative, which indicates that the function  $f$  is not convex  
7   and consequently does not contain a global minimum within the subbox;  
8              wherein applying term consistency to the set of relations involves applying  
9   term consistency to each inequality  $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ .

1            8.        The method of claim 7, wherein applying box consistency to the  
2   set of relations involves applying box consistency to each inequality  
3    $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ .

1            9.        The method of claim 1,  
2   wherein performing the interval Newton step involves,  
3              computing the Jacobian  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  of the gradient  $\mathbf{g}$  evaluated  
4   as a function of a point  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ,

5                    computing an approximate inverse **B** of the center of  
6                    **J(x,X)**, and  
7                    using the approximate inverse **B** to analytically determine  
8                    the system **Bg(x)**, wherein **g(x)** is the gradient of the function  $f(\mathbf{x})$ ,  
9                    and wherein **g(x)** includes components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ ); and  
10                    wherein applying term consistency to the set of relations involves applying  
11                    term consistency to each component  $(\mathbf{B}\mathbf{g}(\mathbf{x}))_i = 0$  ( $i=1, \dots, n$ ) for each variable  
12                     $x_i$  ( $i=1, \dots, n$ ) over the subbox **X**.

1                    10.    The method of claim 9, wherein applying box consistency to the  
2                    set of relations involves applying box consistency to each component  $(\mathbf{B}\mathbf{g}(\mathbf{x}))_i = 0$   
3                    ( $i=1, \dots, n$ ) for each variable  $x_i$  ( $i=1, \dots, n$ ) over the subbox **X**.

1                    11.    The method of claim 1, further comprising terminating attempts to  
2                    further reduce the subbox **X** when:  
3                    the width of **X** is less than a first threshold value; and  
4                    the magnitude of  $f(\mathbf{X})$  is less than a second threshold value.

1                    12.    The method of claim 11, wherein performing the interval Newton  
2                    step involves:  
3                    computing **J(x,X)**, wherein **J(x,X)** is the Jacobian of the function **f**  
4                    evaluated as a function of **x** over the subbox **X**; and  
5                    determining if **J(x,X)** is regular as a byproduct of solving for the subbox **Y**  
6                    that contains values of **y** that satisfy  $\mathbf{M}(\mathbf{x},\mathbf{X})(\mathbf{y}-\mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where  
7                     $\mathbf{M}(\mathbf{x},\mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x},\mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and **B** is an approximate inverse of the center of  
8                    **J(x,X)**.

1           13.     A computer-readable storage medium storing instructions that  
2     when executed by a computer cause the computer to perform a method for using a  
3     computer system to solve an unconstrained interval global optimization problem  
4     specified by a function  $f$ , wherein  $f$  is a scalar function of a vector  
5      $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$ , the method comprising:  
6           receiving a representation of the function  $f$  at the computer system;  
7           storing the representation in a memory within the computer system; and  
8           performing an interval global optimization process to compute guaranteed  
9     bounds on a globally minimum value of the function  $f(\mathbf{x})$  over a subbox  $\mathbf{X}$ ;  
10    wherein performing the interval global optimization process involves,  
11           applying term consistency to a set of relations associated  
12           with the function  $f$  over the subbox  $\mathbf{X}$ , and excluding any portion of  
13           the subbox  $\mathbf{X}$  that violates any of these relations,  
14           applying box consistency to the set of relations associated  
15           with the function  $f$  over the subbox  $\mathbf{X}$ , and excluding any portion of  
16           the subbox  $\mathbf{X}$  that violates any of these relations, and  
17           performing an interval Newton step on the subbox  $\mathbf{X}$  to  
18           produce a resulting subbox  $\mathbf{Y}$ , wherein the point of expansion of  
19           the interval Newton step is a point  $\mathbf{x}$  within the subbox  $\mathbf{X}$ , and  
20           wherein performing the interval Newton step involves evaluating  
21           the gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$  using interval arithmetic.

1           14.     The computer-readable storage medium of claim 13, wherein  
2     applying term consistency involves:  
3           symbolically manipulating an equation within the computer system to  
4     solve for a term  $g(x_j)$ , thereby producing a modified equation  $g(x_j) = h(\mathbf{x})$ , wherein  
5     the term  $g(x_j)$  can be analytically inverted to produce an inverse function  $g^{-1}(y)$ ;

6 substituting the subbox  $\mathbf{X}$  into the modified equation to produce the  
7 equation  $g(X'_j) = h(\mathbf{X})$ ;  
8 solving for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and  
9 intersecting  $X'_j$  with the interval  $X_j$  to produce a new subbox  $\mathbf{X}^+$ ;  
10 wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the equation within  
11 the subbox  $\mathbf{X}$ , and wherein the size of the new subbox  $\mathbf{X}^+$  is less than or equal to  
12 the size of the subbox  $\mathbf{X}$ .

1 15. The computer-readable storage medium of claim 13, wherein  
2 performing the interval global optimization process involves:  
3 keeping track of a smallest upper bound  $f\_bar$  of the function  $f(\mathbf{x})$ ;  
4 removing from consideration any subbox  $\mathbf{X}$  for which  $f(\mathbf{X}) > f\_bar$ ; and  
5 wherein applying term consistency to the  $f\_bar$  relation involves applying  
6 term consistency to the inequality  $f(\mathbf{x}) \leq f\_bar$  over the subbox  $\mathbf{X}$ .

1 16. The computer-readable storage medium of claim 15, wherein  
2 applying box consistency to the set of relations involves applying box consistency  
3 to the inequality  $f(\mathbf{x}) \leq f\_bar$  over the subbox  $\mathbf{X}$ .

1 17. The computer-readable storage medium of claim 13, wherein  
2 performing the interval global optimization process involves:  
3 determining the gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  includes  
4 components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ );  
5 removing from consideration any subbox for which any element of  $\mathbf{g}(\mathbf{x})$  is  
6 bounded away from zero, thereby indicating that the subbox does not include a  
7 stationary point of  $f(\mathbf{x})$ ; and

8            wherein applying term consistency to the set of relations involves applying  
9 term consistency to each component  $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$  over the subbox  
10  $\mathbf{X}$ .

1            18.    The computer-readable storage medium of claim 17, wherein  
2 applying box consistency to the set of relations involves applying box consistency  
3 to each component  $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$  over the subbox  $\mathbf{X}$ .

1            19.    The computer-readable storage medium of claim 13, wherein  
2 performing the interval global optimization process involves:  
3            determining diagonal elements  $H_{ii}(\mathbf{x})$  ( $i=1, \dots, n$ ) of the Hessian of the  
4 function  $f(\mathbf{x})$ ;  
5            removing from consideration any subbox for which a diagonal element of  
6 the Hessian is always negative, which indicates that the function  $f$  is not convex  
7 and consequently does not contain a global minimum within the subbox;  
8            wherein applying term consistency to the set of relations involves applying  
9 term consistency to each inequality  $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ .

1            20.    The computer-readable storage medium of claim 19, wherein  
2 applying box consistency to the set of relations involves applying box consistency  
3 to each inequality  $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ .

1            21.    The computer-readable storage medium of claim 13,  
2 wherein performing the interval Newton step involves,  
3            computing the Jacobian  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  of the gradient  $\mathbf{g}$  evaluated  
4 as a function of a point  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ,

5 computing an approximate inverse  $\mathbf{B}$  of the center of  
6  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , and  
7 using the approximate inverse  $\mathbf{B}$  to analytically determine  
8 the system  $\mathbf{B}\mathbf{g}(\mathbf{x})$ , wherein  $\mathbf{g}(\mathbf{x})$  is the gradient of the function  $f(\mathbf{x})$ ,  
9 and wherein  $\mathbf{g}(\mathbf{x})$  includes components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ ); and  
10 wherein applying term consistency to the set of relations involves applying  
11 term consistency to each component  $(\mathbf{B}\mathbf{g}(\mathbf{x}))_i = 0$  ( $i=1, \dots, n$ ) for each variable  
12  $x_i$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ .

1 22. The computer-readable storage medium of claim 21, wherein  
2 applying box consistency to the set of relations involves applying box consistency  
3 to each component  $(\mathbf{B}\mathbf{g}(\mathbf{x}))_i = 0$  ( $i=1, \dots, n$ ) for each variable  $x_i$  ( $i=1, \dots, n$ ) over the  
4 subbox  $\mathbf{X}$ .

1 23. The computer-readable storage medium of claim 13, wherein the  
2 method further comprises terminating attempts to further reduce the subbox  $\mathbf{X}$   
3 when:  
4 the width of  $\mathbf{X}$  is less than a first threshold value; and  
5 the magnitude of  $f(\mathbf{X})$  is less than a second threshold value.

1 24. The computer-readable storage medium of claim 13, wherein  
2 performing the interval Newton step involves:  
3 computing  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$   
4 evaluated as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and  
5 determining if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
6 that contains values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where



1  $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and  $\mathbf{B}$  is an approximate inverse of the center of  
2  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ .

1 25. An apparatus that solves an unconstrained interval global  
2 optimization problem specified by a function  $f$ , wherein  $f$  is a scalar function of a  
3 vector  $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$ , the apparatus comprising:  
4 a receiving mechanism that is configured to receive a representation of the  
5 function  $f$ ;  
6 a memory for storing the representation; and  
7 an interval global optimization mechanism that is configured to perform  
8 an interval global optimization process to compute guaranteed bounds on a  
9 globally minimum value of the function  $f(\mathbf{x})$  over a subbox  $\mathbf{X}$ ;  
10 a term consistency mechanism within the interval global optimization  
11 mechanism that is configured to apply term consistency to a set of relations  
12 associated with the function  $f$  over the subbox  $\mathbf{X}$ , and to exclude any portion of the  
13 subbox  $\mathbf{X}$  that violates any of these relations;  
14 a box consistency mechanism within the interval global optimization  
15 mechanism that is configured to apply box consistency to the set of relations  
16 associated with the function  $f$  over the subbox  $\mathbf{X}$ , and to exclude any portion of the  
17 subbox  $\mathbf{X}$  that violates any of these relations; and  
18 an interval Newton mechanism within the interval global optimization  
19 mechanism that is configured to perform an interval Newton step on the subbox  $\mathbf{X}$   
20 to produce a resulting subbox  $\mathbf{Y}$ , wherein the point of expansion of the interval  
21 Newton step is a point  $\mathbf{x}$  within the subbox  $\mathbf{X}$ , and wherein performing the interval  
22 Newton step involves evaluating the gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$  using  
23 interval arithmetic.

1           26.     The apparatus of claim 25, wherein the term consistency  
2 mechanism is configured to:  
3           symbolically manipulate an equation to solve for a term  $g(x_j)$ , thereby  
4 producing a modified equation  $g(x_j) = h(\mathbf{x})$ , wherein the term  $g(x_j)$  can be  
5 analytically inverted to produce an inverse function  $g^{-1}(y)$ ;  
6           substitute the subbox  $\mathbf{X}$  into the modified equation to produce the equation  
7  $g(X'_j) = h(\mathbf{X})$ ;  
8           solve for  $X'_j = g^{-1}(h(\mathbf{X}))$ ; and to  
9           intersect  $X'_j$  with the interval  $X_j$  to produce a new subbox  $\mathbf{X}^+$ ;  
10          wherein the new subbox  $\mathbf{X}^+$  contains all solutions of the equation within  
11 the subbox  $\mathbf{X}$ , and wherein the size of the new subbox  $\mathbf{X}^+$  is less than or equal to  
12 the size of the subbox  $\mathbf{X}$ .

1           27.     The apparatus of claim 25,  
2 wherein the interval global optimization mechanism is configured to,  
3           keep track of a smallest upper bound  $f\_bar$  of the function  
4  $f(\mathbf{x})$ , and to  
5           remove from consideration any subbox  $\mathbf{X}$  for which  
6  $f(\mathbf{X}) > f\_bar$ ; and  
7           wherein the term consistency mechanism is configured to apply term  
8 consistency to the inequality  $f(\mathbf{x}) \leq f\_bar$  over the subbox  $\mathbf{X}$ .

1           28.     The apparatus of claim 27, wherein the box consistency  
2 mechanism is configured to apply box consistency to the inequality  $f(\mathbf{x}) \leq f\_bar$   
3 over the subbox  $\mathbf{X}$ .

1           29.     The apparatus of claim 25,

2 wherein the interval global optimization mechanism is configured to,  
3 determine the gradient  $\mathbf{g}(\mathbf{x})$  of the function  $f(\mathbf{x})$ , wherein  
4  $\mathbf{g}(\mathbf{x})$  includes components  $g_i(\mathbf{x})$  ( $i=1, \dots, n$ ), and to  
5 remove from consideration any subbox for which any  
6 element of  $\mathbf{g}(\mathbf{x})$  is bounded away from zero, thereby indicating that  
7 the subbox does not include a stationary point of  $f(\mathbf{x})$ ; and  
8 wherein the term consistency mechanism is configured to apply term  
9 consistency to each component  $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$  over the subbox  $\mathbf{X}$ .

1 30. The apparatus of claim 29, wherein the box consistency  
2 mechanism is configured to apply box consistency to each component  
3  $g_i(\mathbf{x})=0$  ( $i=1, \dots, n$ ) of  $\mathbf{g}(\mathbf{x})=\mathbf{0}$  over the subbox  $\mathbf{X}$ .

1 31. The apparatus of claim 25,  
2 wherein the interval global optimization mechanism is configured to,  
3 determine diagonal elements  $H_{ii}(\mathbf{x})$  ( $i=1, \dots, n$ ) of the  
4 Hessian of the function  $f(\mathbf{x})$ , and to  
5 remove from consideration any subbox for which a  
6 diagonal element of the Hessian is always negative, which  
7 indicates that the function  $f$  is not convex and consequently does  
8 not contain a global minimum within the subbox;  
9 wherein the term consistency mechanism is configured to apply term  
10 consistency to each inequality  $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ .

1 32. The apparatus of claim 31, wherein the box consistency  
2 mechanism is configured to apply box consistency to each inequality  
3  $H_{ii}(\mathbf{x}) \geq 0$  ( $i=1, \dots, n$ ) over the subbox  $\mathbf{X}$ .



- 1           compute  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ , wherein  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is the Jacobian of the function  $\mathbf{f}$  evaluated  
2 as a function of  $\mathbf{x}$  over the subbox  $\mathbf{X}$ ; and to  
3           determine if  $\mathbf{J}(\mathbf{x}, \mathbf{X})$  is regular as a byproduct of solving for the subbox  $\mathbf{Y}$   
4 that contains values of  $\mathbf{y}$  that satisfy  $\mathbf{M}(\mathbf{x}, \mathbf{X})(\mathbf{y} - \mathbf{x}) = \mathbf{r}(\mathbf{x})$ , where  
5  $\mathbf{M}(\mathbf{x}, \mathbf{X}) = \mathbf{B}\mathbf{J}(\mathbf{x}, \mathbf{X})$ ,  $\mathbf{r}(\mathbf{x}) = -\mathbf{B}\mathbf{f}(\mathbf{x})$ , and  $\mathbf{B}$  is an approximate inverse of the center of  
6  $\mathbf{J}(\mathbf{x}, \mathbf{X})$ .